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Asymptotic Properties of Positive Summation–Integral Operators

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1. INTRODUCTION

The Bernstein operator B_n defined by

$$B_n(f)(x) = \sum_{k=0}^{n} f\left(\frac{k}{n}\right) b_{nk}(x),$$

where $b_{nk}(x) = \binom{n}{k} x^k(1-x)^{n-k}$, gives explicit polynomial approximants to any continuous function f on $[0, 1]$. To take care of the larger class $L_1[0, 1]$ of functions f integrable on $[0, 1]$, the Kantorovich operator K_n defined by

$$K_n(f)(x) = \sum_{k=0}^{n} b_{nk}(x) \int_0^1 (n+1)\, \chi_{nk}(t)\, f(t)\, dt,$$

where χ_{nk} is the characteristic function of the interval $[k/(n+1), (k+1)/(n+1)]$, can be used. In the latter case, the kernel $\{(n+1)\,\chi_{nk}\}$ plays the role of "smoothing" the data function f. In general, the Bernstein operator B_n can be generalized to a positive linear operator L_n defined by

$$L_n(f)(x) = \sum_{k=0}^{n} f(x_{nk})\, \lambda_{nk}(x), \tag{1}$$

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where $a \leqslant x_{n0} < \cdots < x_{nk} \leqslant b$ and λ_{nk}'s are nonnegative functions in $PC[a, b]$, the class of all piecewise continuous functions on $[a, b]$, satisfying the Korovkin condition

$$(\mathbb{K}) \qquad \|L_n(\phi_i) - \phi_i\|_{[a,b]} \to 0, \qquad i = 1, 2, 3,$$

where $\phi_1(x) = 1$, $\phi_2(x) = x$, and $\phi_3(x) = x^2$.

Here and throughout, $\| \ \|_F$ denotes, as usual, the supremum norm over F. In some situations, such as the Landau polynomial operator, the supremum must be taken over any closed subset of the open interval (a, b), and the approximation of $f \in C[a, b]$ by $L_n(f)$ would be uniform on compact subsets of (a, b), instead of the whole set $[a, b]$ implied by condition $(\mathbb{K})$ stated above.

In order to generalize L_n in (1) to a summation–integral operator Λ_n, the family $\{(n+1)\chi_{nk}\}$ used by Kantorovich may be replaced by a family $\{\omega_{nk}\}$ of not necessarily positive functions in $L_1(a, b)$, satisfying the normalization property

$$\int_a^b \omega_{nk}(t)\, dt = 1, \qquad k = 0, ..., n \qquad \text{and} \qquad n = 1, 2, \ldots . \tag{2}$$

The operator Λ_n now has kernel

$$s_n(x, t) = \sum_{k=0}^{n} \lambda_{nk}(x)\, \omega_{nk}(t)$$

and takes on the form

$$\begin{aligned} \Lambda_n(f)(x) &= \int_a^b s_n(x, t)\, f(t)\, dt \\ &= \sum_{k=0}^{n} \lambda_{nk}(x) \int_a^b \omega_{nk}(t)\, f(t)\, dt. \end{aligned} \tag{3}$$

This summation–integral operator can also be called a linear smoothing operator, since the integral kernel $\{\omega_{nk}\}$ indeed smooths the data function f. The general class of operators Λ_n includes the ones considered in the literature (cf. [1–3] and [5–13]). In [4], we gave a fairly thorough treatment of the generalization of Λ_n to functions of s variables and obtained both necessary and sufficient conditions on the kernel $\{\omega_{nk}\}$ that guarantees convergence of $\{\Lambda_n(f)\}$ to f.

In this paper, we will go one step further and study the asymptotic properties of $\{\Lambda_n\}$. Hence, somewhat stronger conditions on $\{\omega_{nk}\}$ must be imposed and we will state these conditions in the next section. To simplify the technicality on the conditions of $\{\omega_{nk}\}$, we only consider the special case $\omega_{nk} \geqslant 0$. Our main result is that under these conditions on

$\{\omega_{nk}\}$, certain asymptotic properties of the summation operators $\{L_n\}$ are preserved by the summation–integral operators $\{\Lambda_n\}$. Of course, when Voronovskaja-type formulas are considered, the "limits" of $\{\Lambda_n\}$ not only depend on $\{L_n\}$ but also on $\{\omega_{nk}\}$. We will give explicit formulation of these formulas for the classical kernels $\{\lambda_{nk}\}$ and $\{\omega_{nk}\}$. In particular, for the Kantorovich operator, where $\lambda_{nk} = b_{nk}$ and $\omega_{nk} = (n+1)\,\chi_{nk}$, our result shows that the Voronovskaja formula obtained by Wafi, Habib, and Kahn [14] is incorrect.

2. Main Results

The asymptotic properties we are going to discuss in this paper can be classified into two categories: (i) Voronovskaja formulas, and (ii) asymptotic shape-preserving properties.

We will use the standard notation,

$$\Delta_h^m f(x), \qquad h>0,$$

to designate the mth order forward difference of the function f at x with increment h; that is,

$$\begin{aligned}\Delta_h^m f(x) &= \Delta_h^{m-1}(\Delta_h f)(x)\\ &= \Delta_h^{m-1}(f(x+h)-f(x)).\end{aligned}$$

Definition. *A sequence of linear operators $\{P_n\}$ is said to have the mth order asymptotic shape-preserving property if $n^m \Delta_{n^{-1}}^m P_n(f)(x) \to f^{(m)}(x)$, $x \in [a,b]$, for every $f \in C^m[a,b]$.*

We first recall the following Voronovskaja formula for the Bernstein operator B_n due to Lorentz [8]. Let $f \in C^m[0,1]$ where m is a positive even integer. Then

$$B_n(f)(x) - f(x) = \sum_{j=1}^{m/2} \sum_{i=1}^{m} \frac{d_{j,i}(x)}{n^j} f^{(i)}(x) + o(n^{-m/2}), \tag{4}$$

where, for $j = i - [i/2]$, $[y]$ denotes the integer part of y,

$$d_{j,i}(x) = \frac{1}{i!} n^{-[i/2]} \sum_{k=0}^{n} (k-nx)^i \binom{n}{k} x^k (1-x)^{n-k},$$

and $d_{j,i}(x) \equiv 0$ for $j \neq i - [i/2]$. In particular, if $m=2$, then

$$B_n(f)(x) - f(x) = \frac{1}{2n} x(1-x) f''(x) + o\left(\frac{1}{n}\right). \tag{5}$$

Also, the following asymptotic shape-preserving result is implicitly contained in [8].

$$n^2 \Delta_{n^{-1}}^2 B_n(f)(x) \to f''(x), \qquad x \in [0, 1], \tag{6}$$

for every $f \in C^2[0, 1]$.

It is not surprising that other classical positive linear operators L_n as defined in (1) may have analogous asymptotic properties as B_n, but it is not clear if such properties are preserved by Λ_n defined in (3). The first paper that dealt with this problem is [14], where the Kantorovich operator K_n was considered, and it was claimed that, for $f \in C^2[0, 1]$,

$$K_n(f) - f(x) = \frac{1}{8n} f''(x) + o\left(\frac{1}{n}\right), \qquad x \in [0, 1]. \tag{7}$$

We will see in this paper that this formula is incorrect.

In general, we will assume that the operators L_n satisfy certain asymptotic properties analogous to (4) and a formula more general than (6) and identify conditions on the smoothing kernel $\{\omega_{nk}\}$ so that the summation–integral operators Λ_n also enjoy these asymptotic properties.

Throughout this paper, we will consider the following assumptions on $\{L_n\}$ and $\{\omega_{nk}\}$:

(A) Let l and m be positive integers with $l < m$ and m even.

(i) $$L_n(f)(x) - f(x) = \sum_{j=1}^{l} \sum_{i=1}^{m} \frac{g_{j,i}(x)}{n^j} f^{(i)}(x) + o(n^{-l}),$$

$x \in [a, b]$, for all $f \in C^m[a, b]$, where $g_{j,i}$, $i = 1, ..., m$, and $j = 1, ..., l$, are continuous functions on $[a, b]$,

(ii) For all $f \in C^l[a, b]$,

$$n^l \Delta_{n^{-1}}^l L_n(f)(x) \to f^{(l)}(x), \qquad x \in [a, b].$$

(B) Let l and m be as above and $\omega_{nk} \geq 0$.

(i) There exist functions $\alpha_{j,i} \in C[a, b]$, $i = 1, ..., m$, and $j = 1, ..., l$, such that

$$\max_{0 \leq k \leq n} \left| \int_a^b \omega_{nk}(t)(t - x_{nk})^i \, dt - \sum_{j=1}^{l} \frac{\alpha_{j,i}(x_{nk})}{n^j} \right| = o(n^{-l})$$

for each $i = 1, ..., m$.

$$\text{(ii)} \qquad \max_{0 \leqslant k \leqslant n} \int_{|t - x_{nk}| > \delta} \omega_{nk}(t)\, dt = o(n^{-l})$$

for every $\delta > 0$.

We have the following results.

THEOREM 1. *Let conditions* (Ai) *and* (B) *be satisfied. Then*

$$\begin{aligned} A_n(f)(x) - f(x) = \sum_{j=1}^{l} \frac{1}{n^j} \Bigg\{ & \sum_{i=1}^{m} \Bigg[\sum_{s=1}^{j} \sum_{v=0}^{i-1} \sum_{u=v}^{m} \binom{u}{v} \\ & \times \frac{g_{j-s,u}(x)}{(i-v)!} \alpha_{s,i-v}^{(u-v)}(x) + g_{j,i}(x) \Bigg] f^{(i)}(x) \\ & + \sum_{i=m+1}^{2m} \sum_{s=1}^{j} \sum_{v=i-m}^{m} \sum_{u=v}^{m} \binom{u}{v} \\ & \times \frac{g_{j-s,u}(x)}{(i-v)!} \alpha_{s,i-v}^{(u-v)}(x)\, f^{(i)}(x) \Bigg\} + o(n^{-l}), \end{aligned} \tag{8}$$

$x \in [a, b]$, *for all* $f \in C^m[a, b]$, *where* $g_{00}(x) =: 1$, *and for* $(i, j) \neq 0$, $g_{i,j}(x) =: 0$ *if* $i > m$, $i = 0$, *or* $j = 0$. *In particular,*

$$A_n(f)(x) - f(x) = \sum_{i=1}^{2} \left(\frac{1}{i!} \alpha_{1,i}(x) + g_{1,i}(x) \right) f^{(i)}(x) + o(n^{-1}) \tag{9}$$

for all $f \in C^2[a, b]$.

THEOREM 2. *Let conditions* (Aii) *and* (B) *be satisfied. Then*

$$n^l \Delta_{n-1}^l L_n(f)(x) \to f^{(l)}(x),$$

$x \in [a, b]$, *for all* $f \in C^l[a, b]$.

3. Proof of Results

Let $f \in C^m[a, b]$ and write

$$\begin{aligned} f(t) = f(x_{nk}) + f'(x_{nk})(t - x_{nk}) + \cdots + f^{(m)}(x_{nk}) \frac{(t - x_{nk})^m}{m!} \\ + \eta_{nk}(t - x_{nk}) \frac{(t - x_{nk})^m}{m!}, \end{aligned} \tag{10}$$

where $\eta_{nk}(t-x_{nk}) \to 0$ as $t \to x_{nk}$ uniformly in $k \in [0, ..., n]$. Hence, we may write

$$\Lambda_n(f)(x) = L_n(f) + \sum_{i=1}^{m} \sum_{k=0}^{n} \lambda_{nk}(x) f^{(i)}(x_{nk}) \times \frac{1}{i!} \int_a^b \omega_{nk}(t)(t-x_{nk})^i \, dt + R_n(x),$$

where

$$R_n(x) = \sum_{k=0}^{n} \lambda_{nk}(x) \frac{1}{n!} \int_a^b \omega_{nk}(t)(t-x_{nk})^m \eta_{nk}(t-x_{nk}) \, dt.$$

Now, using the convention that an empty sum is zero and the notation $g_{00} = 1$, we have

$$\begin{aligned}
&\Lambda_n(f)(x) - L_n(f)(x) - R_n(x) \\
&= \sum_{i=1}^{m} \sum_{k=0}^{n} \lambda_{nk}(x) f^{(i)}(x_{nk}) \frac{1}{i!} \left(\sum_{j=1}^{l} \frac{\alpha_{j,i}(x_{nk})}{n^j} + o(n^{-l}) \right) \\
&= \sum_{j=1}^{l} \sum_{i=1}^{m} \frac{1}{i!\, n^j} \left(\sum_{k=0}^{n} \lambda_{nk}(x) f^{(i)}(x_{nk}) \alpha_{j,i}(x_{nk}) \right) + o(n^{-l}) \\
&= \sum_{j=1}^{l} \sum_{i=1}^{m} \frac{1}{i!\, n^j} \Bigg(f^{(i)}(x) \alpha_{j,i}(x) \\
&\quad + \sum_{s=1}^{l-j} \sum_{u=1}^{m} \frac{g_{s,u}(x)}{n^s} [f^{(i)}(x) \alpha_{j,i}(x)]^{(u)} + o(n^{-l+j}) \Bigg) + o(n^{-l}) \\
&= \sum_{j=1}^{l} \sum_{i=1}^{m} \sum_{s=0}^{l-j} \sum_{u=0}^{m} \frac{g_{s,u}(x)}{i!\, n^{s+j}} [f^{(i)}(x) \alpha_{j,i}(x)]^{(u)} + o(n^{-l}) \\
&= \sum_{j=1}^{l} \sum_{i=1}^{m} \sum_{s=0}^{l-j} \sum_{u=0}^{m} \sum_{v=0}^{u} \binom{u}{v} \frac{g_{s,u}(x)}{i!\, n^{s+j}} \alpha_{j,i}^{(u-v)}(x) f^{(i+v)}(x) + o(n^{-l}) \\
&= \sum_{j=1}^{l} \sum_{s=1}^{j} \Bigg[\sum_{i=1}^{m} \sum_{v=0}^{i-1} \sum_{u=v}^{m} \binom{u}{v} \frac{g_{j-s,u}(x)}{(i-v)!\, n^j} \alpha_{s,i-v}^{(u-v)}(x) f^{(i)}(x) \\
&\quad + \sum_{i=m+1}^{2m} \sum_{v=i-m}^{m} \sum_{u=v}^{m} \binom{u}{v} \frac{g_{j-s,u}(x)}{(i-v)!\, n^j} \alpha_{s,i-v}^{(u-v)}(x) f^{(i)}(x) \Bigg] + o(n^{-l}),
\end{aligned}$$

where $\binom{0}{0} =: 1$. Hence, it is sufficient to show that $R_n(x) = o(n^{-l})$. Let $0 < \varepsilon_n < n^{-l}$, and choose $\delta_n > 0$ such that

$$\max_{0 \leqslant k \leqslant n} |\eta_{nk}(t-x_{nk})| \leqslant \varepsilon_n \tag{11}$$

for all $t \in [a, b]$ with $|t - x_{nk}| < \delta_n$. On the other hand, the family $\{(t - x_{nk})^m \eta_{nk}(t - x_{nk})\}$ is uniformly bounded by some constant M. Hence, writing $R_n = R_{n1} + R_{n2}$ where R_{n1} is the portion where the integral is taken over $|t - x_{nk}| \leqslant \delta_n$ and R_{n2} the remaining portion, we have

$$\begin{aligned}|R_{n1}(x)| &\leqslant \frac{1}{m!} \varepsilon_n \sum_{k=0}^{n} \lambda_{nk}(x) \int_a^b \omega_{nk}(t) |t - x_{nk}|^m \, dt \\ &\leqslant \frac{n^{-l}}{m!} \sum_{k=0}^{n} \lambda_{nk}(x) \left[\sum_{k=1}^{l} \frac{\alpha_{j,i}(x_{nk})}{n^j} + o(n^{-l}) \right] \\ &= o(n^{-l})\end{aligned}$$

and

$$\begin{aligned}|R_{n2}(x)| &\leqslant \sum_{k=0}^{n} \lambda_{nk}(x) \frac{1}{m!} \int_{|t - x_{nk}| > \delta_n} \omega_{nk}(t) \, |(t - x_{nk})^m \eta_{nk}(t - x_{nk})| \, dt \\ &\leqslant \frac{M}{m!} \sum_{k=0}^{n} \lambda_{nk}(x) \int_{|t - x_{nk}| > \delta_n} \omega_{nk}(t) \, dt = o(n^{-l}).\end{aligned}$$

This completes the proof of Theorem 1.

To prove Theorem 2, we again use the Taylor expansion (10) with the error estimate (11). Then we have

$$n^l \Delta_{n-1}^l \Lambda_n(f)(x) = n^l \Delta_{n-1}^l L_n(f)(x) + S_n(x) + T_n(x),$$

where

$$S_n(x) = \sum_{i=1}^{m} \sum_{k=0}^{n} n^l \Delta_{n-1}^l \lambda_{nk}(x) \int_a^b \omega_{nk}(t) \frac{1}{i!} (t - x_{nk})^i \, dt$$

and

$$T_n(x) = \frac{1}{m!} \sum_{k=0}^{n} n^l \Delta_{n-1}^l \lambda_{nk}(x) \int_a^b \omega_{nk}(t) \, \eta_{nk}(t - x_{nk})(t - x_{nk})^m \, dt.$$

In view of the hypothesis (Aii), it is sufficient to prove that both $S_n(x)$ and $T_n(x)$ are $o(1)$. Now,

$$\begin{aligned}S_n(x) &= \sum_{i=0}^{m} \frac{1}{i!} \sum_{k=0}^{n} n^l \Delta_{n-1}^l \lambda_{nk}(x) \left[\sum_{j=1}^{l} \frac{\alpha_{j,i}(x_{nk})}{n^j} + o(n^{-l}) \right] \\ &= \sum_{i=1}^{m} \frac{1}{i!} \left[\sum_{j=1}^{l} \left(\frac{1}{n^j} n^l \Delta_{n-1}^l L_n(\alpha_{j,i})(x) \right) \right. \\ &\qquad \left. + \, o(n^{-l}) \, n^l \Delta_{n-1}^l L_n(1)(x) \right] = o\left(\frac{1}{n}\right).\end{aligned}$$

To estimate $T_n(x)$, we again write $T_n = T_{n1} + T_{n2}$ where T_{n1} takes care of the integral over $|t - x_{nk}| \leqslant \delta_n$ and T_{n2} the remaining portion. Then

$$\begin{aligned}|T_{n1}(x)| &\leqslant \frac{1}{m!} \varepsilon_n \sum_{k=0}^{n} n^l \, |\Delta_{n-1}^l \, \lambda_{nk}(x)| \int_a^b \omega_{nk}(t) \, |t - x_{nk}|^m \, dt \\ &\leqslant \frac{1}{m!} \sum_{k=0}^{n} |\Delta_{n-1}^l \, \lambda_{nk}(x)| \left[\sum_{j=1}^{l} \frac{\alpha_{j,i}(x_{nk})}{n^j} + o(n^{-l}) \right] \\ &= o(n^{-1})\end{aligned}$$

and

$$\begin{aligned}|T_{n2}(x)| &\leqslant \frac{1}{m!} \sum_{k=0}^{n} n^l \, |\Delta_{n-1}^l \, \lambda_{nk}(x)| \int_{|t - x_{nk}| > \delta_n} \omega_{nk}(t) \, |(t - x_{nk})^m \, \eta_{nk}(t - x_{nk})| \, dt \\ &= \frac{M}{m!} \sum_{k=0}^{n} n^l \, |\Delta_{n-1}^l \, \lambda_{nk}(x)| \int_{|t - x_{nk}| > \delta_n} \omega_{nk}(t) \, dt \\ &= o(1).\end{aligned}$$

This completes the proof of Theorem 2.

4. Examples

In order that $\{\Lambda_n\}$ preserves the asymptotic properties (Ai) or (Aii) of $\{L_n\}$, the kernel $\{\omega_{nk}\}$ must satisfy both Conditions (Bi) and (Bii). We have shown that all the classical kernels of Bernstein, Kantorovich, Landau, Szasz-Mirakjan, and Rappoport satisfy these conditions, and in the following, we list the functions $\alpha_{j,i}(x)$ in Condition (Bi).

(1) Bernstein. For the kernel $\omega_{nk}(t) = (n+1)\binom{n}{k} t^k (1-t)^{n-k}$, with $x_{nk} = k/n$ on $[0, 1]$, we have

$$\begin{aligned}\alpha_{j,i}(x) = x^i \sum_{v=j}^{i} \binom{i}{v} (-1)^v &\left(\sum_{i-v+2 \leqslant p_1 < \cdots < p_j \leqslant i+1} p_1 \cdots p_j \right) \\ &+ \cdots + x^{i-j+u} \sum_{v=u}^{i-j+u} \binom{i}{v} (-1)^v \left(\sum_{1 \leqslant p_1 < \cdots < p_{j-u} \leqslant i-v} p_1 \cdots p_{j-u} \right) \\ &\times \left(\sum_{i-v+2 \leqslant p_1 < \cdots < p_u \leqslant i+1} p_1 \cdots p_u \right) \\ &+ \cdots + x^{i-j} \sum_{v=0}^{i-j} \binom{i}{v} (-1)^v \left(\sum_{1 \leqslant p_1 < \cdots < p_j \leqslant i-v} p_1 \cdots p_j \right).\end{aligned}$$

(2) Kantorovich. For the kernel $\omega_{nk}(t)=(n+1)\,\chi_{nk}(t)$, with $x_{nk}=k/n$ on $[0, 1]$, we have

$$\alpha_{j,i}(x)=\begin{cases}[(1-x)^{i+1}-(-x)^{i+1}]/(i+1), & \text{for } j=i\\ 0, & \text{for } j\neq i.\end{cases}$$

(3) Modified Kantorovich. For the kernel

$$\omega_{nk}(t)=\begin{cases}(2n+1)/2\pi, & \text{for } 2\pi k/(2n+1)<t\leqslant 2\pi(k+1)/(2n+1)\\ 0, & \text{otherwise}\end{cases}$$

with $x_{nk}=2\pi k/(2n+1)$ on $[0, 2\pi]$, we have

$$\alpha_{j,i}(x)=\begin{cases}\pi^i/(i+1), & \text{for } j=i\\ 0, & \text{for } j\neq i.\end{cases}$$

(4) Landau. For the kernel

$$\omega_{nk}(t)=\left(n\sum_{k=-n}^{n}\left[1-\left(\frac{k}{n}\right)^2\right]^n\right)^{-1}\left[1-\left(t-\frac{k}{n}\right)^2\right]^n\sim\sqrt{\frac{n}{\pi}}\left[1-\left(t-\frac{k}{n}\right)^2\right]^n$$

with $x_{nk}=k/n$ on $(0, 1)$, we have

$$\alpha_{j,i}(x)=0, \qquad \text{for all } j=1, ..., l \text{ and odd } i$$

$$\alpha_{j,i}(x)=\begin{cases}(i-1)!\,2^{-i+1}/(i/2-1)!, & \text{for } j=i/2 \text{ and even } i\\ 0, & \text{otherwise.}\end{cases}$$

(5) Szasz–Mirakjan. For the kernel $\omega_{nk}(t)=(n/k!)(nt)^k\,e^{-nt}$ with $x_{nk}=k/n$ on $[0, 1]$, we have

$$\alpha_{j,i}(x)=x^{i-j}\sum_{u=j}^{i}\binom{j}{u}(-1)^{i-u}\left(\sum_{1\leqslant p_1<\cdots<p_j\leqslant u}p_1\cdots p_j\right).$$

(6) Rappoport. For the kernel

$$\frac{2n+1}{2\pi}\cdot\frac{2^{2n}(n!)^2}{(2n+1)!}\cos^{2n}\left(\frac{t-x_{nk}}{2}\right)$$

with $x_{nk}=2\pi k/(2n+1)$ on $[0, 2\pi]$, we have

$$\alpha_{j,i}(x)=0, \qquad \text{for all } j=1, ..., l \text{ and odd } i$$

$$\alpha_{j,i}(x)=\begin{cases}(i-1)!\,2^{-i+3}/(i/2-1)!, & \text{for } j=i/2 \text{ and even } i\\ 0, & \text{otherwise.}\end{cases}$$

To form the operator Λ_n, it is perfectly free to choose any λ_{nk} and any ω_{nk} as long as they have the same $\{x_{nk}\}$ and are defined on the same interval. For example, Kantorovich used b_{nk} for λ_{nk} and we may call the corresponding Λ_n Bernstein–Kantorovich operators. Similarly, Bernstein–Bernstein and Ssazs–Mirakjan–Kantorovich (summation–integral) operators have been considered in the literature (cf. [2, 10, 12]).

The most difficult one to verify is probably the Bernstein kernel. So we will only discuss this case. To verify condition (Bii), we recall the estimate

$$b_{nk}(x) \leqslant \sum_{|(k/n)-x| \geqslant n^{-\alpha}} b_{nk}(x) \leqslant Cn^{-l-1}$$

which can be found in Lorentz [8, p. 15], where $0<\alpha<2^{-1}$ is arbitrary, l any positive integer, and C depends only on α and l. Hence

$$n \int_{|t-(k/n)| \geqslant n^{-\alpha}} b_{nk}(t)\, dt \leqslant Cn^{-l}$$

which implies (Bii). To derive the functions $\alpha_{j,i}(x)$ in (1), we note that

$$\begin{aligned}
&\int_0^1 (n+1)\, b_{nk}(t)\left(t-\frac{k}{n}\right)^i dt \\
&= (n+1)\binom{n}{k} \sum_{v=0}^{i} \binom{i}{v}\left(-\frac{k}{n}\right)^v \frac{\Gamma(k+i-v+1)\Gamma(n-k+1)}{\Gamma(n+i-v+2)} \\
&= \frac{(n+1)!}{(n-k)!\,k!} \sum_{v=0}^{i} \binom{i}{v}\left(-\frac{k}{n}\right)^v \frac{(k+i-v)!\,(n-k)!}{(n+i-v+1)!} \\
&= \sum_{v=0}^{i} \binom{i}{v}\left(-\frac{k}{n}\right)^v \left(\frac{k}{n}+\frac{i-v}{n}\right)\cdots\left(\frac{k}{n}+\frac{1}{n}\right) \frac{n^{i-v}}{(n+i-v+1)\cdots(n+2)} \\
&= \frac{1}{(n+i+1)\cdots(n+2)} \sum_{v=0}^{i} \binom{i}{v}\left(-\frac{k}{n}\right)^v \\
&\quad \times \left(\frac{k}{n}+\frac{i-v}{n}\right)\cdots\left(\frac{k}{n}+\frac{1}{n}\right) n^{i-v}(n+i+1)\cdots(n+i-v+2) \\
&= \frac{1}{n}\left[x^i \sum_{v=1}^{i} \binom{i}{v}(-1)^v \left(\sum_{p_1=i-v+2}^{i+1} p_1\right) + x^{i-1} \sum_{v=0}^{i-1} \binom{i}{v}(-1)^v \left(\sum_{p_1=1}^{i-v} p_1\right)\right] \\
&\quad + \frac{1}{n^2}\left[x^i \sum_{v=2}^{i} \binom{i}{v}(-1)^v \left(\sum_{i+2-v \leqslant p_1 < p_2 \leqslant i+1} p_1 p_2\right)\right. \\
&\quad + x^{i-1} \sum_{v=1}^{i-1} \binom{i}{v}(-1)^v \left(\sum_{p_1=1}^{i-v} p_1\right)\left(\sum_{p_2=i+2-v}^{i+1} p_2\right) \\
&\quad \left. + x^{i-2} \sum_{v=0}^{i-2} \binom{i}{v}(-1)^v \left(\sum_{1 \leqslant p_1 < p_2 \leqslant i-v} p_1 p_2\right)\right]
\end{aligned}$$

$$+\cdots+\frac{1}{n^l}\left[x^i\sum_{v=l}^{i}\binom{i}{l}(-1)^v\left(\sum_{i-v+2\leqslant p_1<\cdots<p_l\leqslant i+1}p_1\cdots p_l\right)\right.$$

$$+\cdots+x^{i-l+u}\sum_{v=u}^{i-l+u}\binom{i}{v}(-1)^v\left(\sum_{1\leqslant p_1<\cdots<p_{l-u}\leqslant i-v}p_1\cdots p_{l-u}\right)$$

$$\times\left(\sum_{i-v+2\leqslant p_1<\cdots<p_u\leqslant i+1}p_1\cdots p_u\right)+\cdots+x^{i-l}\sum_{v=0}^{i-l}\binom{i}{v}(-1)^v$$

$$\left.\times\left(\sum_{1\leqslant p_1<\cdots<p_l\leqslant i-v}p_1\cdots p_l\right)\right]+o(n^{-l}),$$

where $x=k/n$. This yields the $\alpha_{j,i}(x)$ in (1).

In the following, we list four simple Voronovskaja formulas:

(i) Bernstein–Bernstein operator. For every $f\in C^2[0,1]$,

$$A_n(f)(x)-f(x)=\frac{1}{n}\left[(1-2x)f'(x)+(x-x^2)f''(x)\right]+o\left(\frac{1}{n}\right).$$

(ii) Bernstein–Kantorovich operator. For every $f\in C^2[0,1]$,

$$K_n(f)(x)-f(x)=\frac{1}{n}\left[\left(\frac{1}{2}-x\right)f'(x)+\frac{x(1-x)}{2}f'(x)\right]+o\left(\frac{1}{n}\right).$$

(iii) Bernstein–Landau operator. For every $f\in C^2[0,1]$,

$$A_n(f)(x)-f(x)=\frac{1}{n}\left[\left(\frac{1}{4}+\frac{x}{2}-\frac{x^2}{2}\right)f''(x)\right]+o\left(\frac{1}{n}\right).$$

(iv) Bernstein–Szász–Mirakjan operator. For every $f\in C^2[0,1]$,

$$A_n(f)(x)-f(x)=\frac{1}{n}\left[f'(x)+\left(x-\frac{x^2}{2}\right)f''(x)\right]+o\left(\frac{1}{n}\right).$$

We finally remark that our Voronovskaja formula on the Bernstein–Kantorovich operator (or simply Kantorovich operator) K_n as shown in (ii) shows that formula (7) obtained by Wafi, Habib, and Khan in [14] is incorrect.

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